Harvard-Smithsonian Center for Astrophysics

Precision Astronomy Group

MEMORANDUM

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To: Distribution

From: R.D. Reasenberg

Subject: Observation density over the sky with the HIPPARCOS observing scheme.

I. Introduction

GAMES is a full-sky astrometric survey instrument with a nominal mission accuracy of 50 μas for bright stars. The mission will determine the photometric magnitude and (five standard) astrometric parameters of about 10^7 stars. It is intended that GAMES will use a version of the HIPPARCOS observing pattern in which the spacecraft spins to allow the observation of stars in two moving directions in the plane perpendicular to the spin axis, and the spin axis slowly precesses around the Sun direction. The angle between the spacecraft spin axis and the Sun direction, ξ (using HIPPARCOS notation), is bounded at the high end by the Sun avoidance requirement and at the low end by the need for observational diversity, which reduces estimator degeneracy. For present purposes, the inertial coordinate system used as a reference for analysis is ecliptic (with no consideration of the motion of the true ecliptic), and references to north and south are to the ecliptic hemispheres.

The first objective of this study is to determine the density of the observations over the celestial sphere as a function of sin(b), where *b* is ecliptic latitude. (Note that sin(b) was chosen because it is proportional to area between the ecliptic and the small circle at *b*.) No consideration is given to the density in ℓ, the ecliptic longitude; in fact, the result should be considered to be averaged over ℓ. The second objective is to determine the range and distribution of observing directions of stars as a function of *b*. This bears on the shape (aspect ratio) of the error ellipse of measured star positions, i.e., the relative uncertainties in estimates of ecliptic latitude and ecliptic longitude.

Below, we consider simulated observations from a single spacecraft look direction. The equations found in Section II determine the observing point and the local direction of the observation as a function of the spacecraft rotation and precession parameters. The numerical results are presented in Section III along with some discussion and interpretation. We conclude that: (1) the distribution of observations over the sky is uneven, but not so uneven as to cause a problem; (2) the distribution of observing directions is non-uniform and depends on the ecliptic latitude, but is unlikely to cause much elongation of the error ellipse for star position.

II. Analysis

My initial approach to addressing the first objective was analytic, but this left me with an Incomplete Elliptic Integral of the First Kind, which is not supported by Mathcad.[1®] I therefore turned to a direct numerical approach, which has the advantage of being able to accommodate additional complexities as they arise.

I had been concerned about the effect of the precession law (for the spacecraft spin axis) on the uniformity of the sky coverage, especially on a possible north-south asymmetry. In particular, I asked: What is the effect on observational uniformity of precessing around the Sun at a rate that is constant (1) in an inertial system, and (2) in a coordinate system that rotates annually, say with the Sun along the x axis? It is now clear that there is no gross effect, as discussed below.

In Fig 1, the reference great circle is the ecliptic, the Sun is at S, and the north ecliptic pole is at Z; the observation point is at V, and its locus is shown as a dashed great circle. Note that, while the precession law may be asymmetric, the spin rate is quite constant. Thus, independent of the location of the spin vector R, observations are conducted equally in the north and south. Further, when the direction of the spin vector is reversed, shifting it between hemispheres, the same set of observations is made.

In comparing the two precession schemes described above, we find that for the second scheme, as compared to the first scheme, the motion of the spin vector around the Sun direction is slower in one hemisphere and correspondingly faster in the other. However, to lowest order, the two schemes afford the same average temporal density for the spin vector at a given (absolute) distance from the ecliptic. Thus, as asserted above, the details of the precession law do not produce a gross effect on the distribution of observation or any effect on the north-south distribution.

Figure 1 shows the geometry of the problem: the spacecraft precession angle is α, and the spin angle is β. In the right spherical triangle SAR, we have

$$\sin a = \sin\xi \cos\alpha \qquad \textbf{(1)}$$

Similarly, working in spherical triangle VZR, we find that

$$\sin b = -\cos a \cos\beta \qquad \textbf{(2)}$$

[1] The reference to a commercial product is for technical communication only, and does not constitute an endorsement of the product.

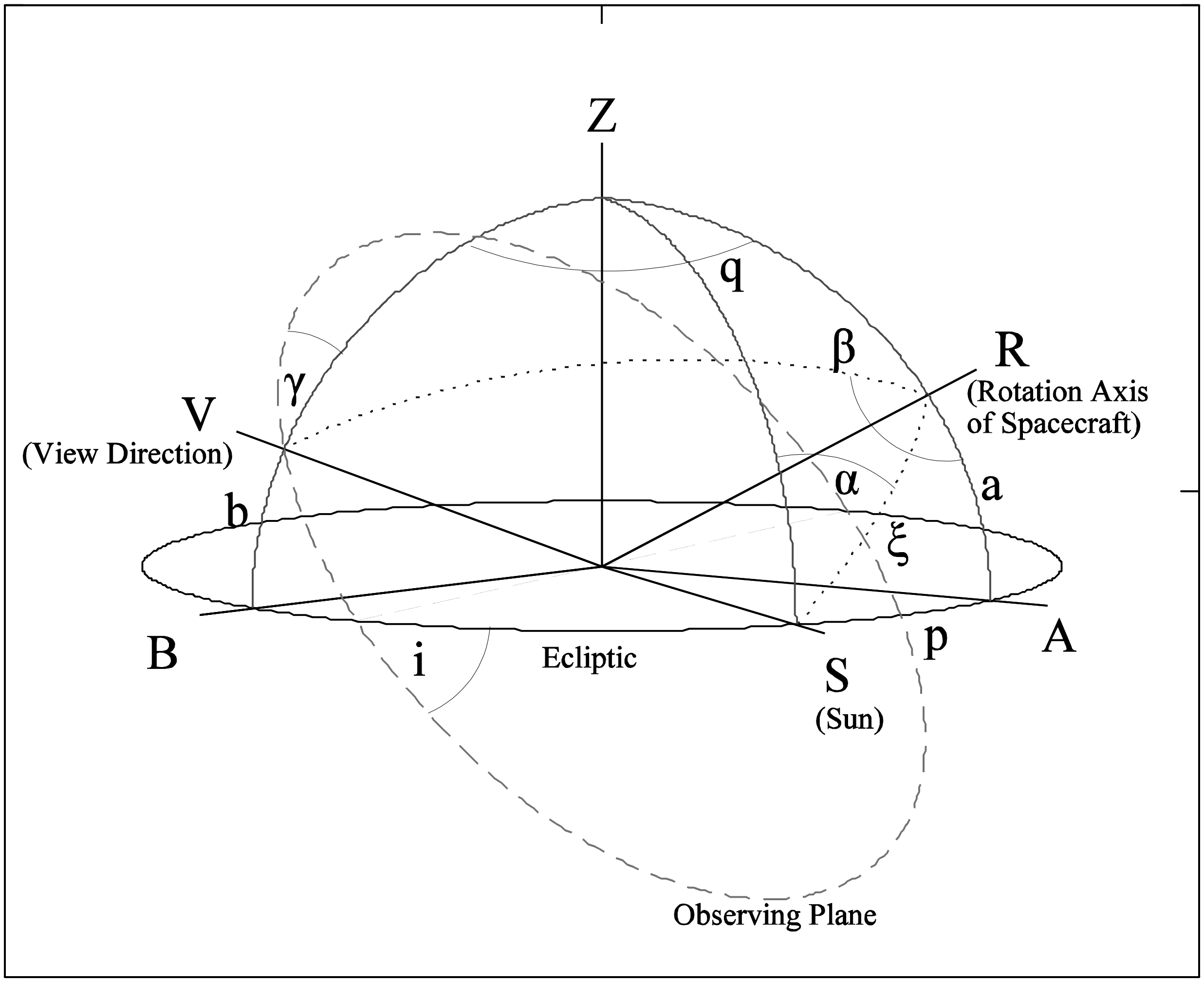

Figure 1. Observation geometry. The spacecraft rotation phase is β, and the precession angle is α. The local direction of observing γ is measured with respect to a great circle through the ecliptic pole. In the drawing, the arc from R to V (the angle between the rotation axis and the view direction) is 90 deg, and ξ, the arc from A to R, is 45 deg. The figure is viewed from 10 deg above the ecliptic.

As discussed in the next section, Mathcad calculated sin(b) as a function of α and β, and histogrammed it. By visualizing V at the ecliptic, and recalling that arc VR is 90 deg, it becomes apparent that $i = \pi/2 - a$.

We next consider the question: For a given observation point, V, from what range of directions is it observed? For this purpose, the observation direction is defined as γ, the angle between the meridian passing through V and the instrument sensitive direction (or observing plane, corresponding to the dashed great circle in Fig 1). From the symmetry of the problem, we know that the range of γ is symmetric around the meridian, $-\tilde{\gamma} < \gamma < \tilde{\gamma}$, and we may make results unique by considering γ such that $|\gamma| < \pi/2$. Again working in VZR, we find that

$$\gamma = \arcsin\left(\frac{\sin a}{\cos b}\right) \quad \textbf{(3)}$$

The principal question is: Do we get a useful distribution in γ for a given observation point, V? To address this question, we may look at the density of observations in the γ-sin(b) plane. Again, I am assuming that we can neglect the detailed dependency on ecliptic longitude, ℓ. If most observations were made in (or near) one direction (i.e., with a single value or small range of γ), then there would be an elongated error ellipse -- an undesirable condition. If a significant fraction of the observations were well distributed (e.g., uniformly, or in two similar size batches with well separated mean directions), then the error ellipse would be acceptably circular.

What are the limits on γ? Since we are considering a fixed observing point (i.e., a fixed value of *b*), Eq 3 shows that γ depends only on *a*. There are two regimes: $\xi + |b| \le \pi/2$ and $\xi + |b| > \pi/2$. In the first, α may assume all values. In the second, α is restricted by a limitation on a + b: $|b + \arcsin(\sin\xi \ \cos\alpha)| \le \pi/2$.

In the first regime

$$\gamma = \arcsin\left(\frac{\sin\xi}{\cos b}\cos\alpha\right) \quad \textbf{(4)}$$

and thus

$$\frac{d\gamma}{d\alpha} = -\frac{\sin\xi}{\cos b\ \cos\gamma}\sin\alpha \quad \textbf{(5)}$$

As α changes uniformly (for *b* fixed and β taking on whatever values are needed -- a non-physical situation that statistically represents the density of observations), γ has an extremum at $\alpha = \{0, \pi\}$. At that point, we find

$$\tilde{\gamma} = \arcsin(\pm \sin\xi / \cos b) \quad \textbf{(6)}$$

This result will gain significance in the next section. In the second regime, there is no such situation.

III. Numerical results and interpretation

Two computational approaches were taken using Mathcad, as discussed below. In both cases, there were a large number of simulated observation points and histograms was formed to summarize the results. Two kinds of histograms were generated. In the first, the binned quantity is the normalized number of observations per step of sin(b). In the second, the binned quantity is the square root of the normalized number of observations. (The square root allows better detail to be seen in both the high and low regions.) In this two-D histogram, the independent quantities are sin(b) and γ. Thus, the first histogram could be generated from the second by summing over γ and renormalizing.

In the first computational approach, the program took even steps over α and β. In the second computational approach, α and β were treated as uniformly distributed random variables over the same ranges, and a Monte Carlo simulation performed. The resulting histograms are essentially the same. Because of the symmetries of the problem, I was able to limit the ranges of the independent variables to $0 < \alpha < 90$ deg, and $0 < \beta < 360$ deg.

Figures 2, 3, and 4 show the histograms for $\xi = 35$, 45, and 55 deg, respectively. In both the one-D and two-D histograms, there is spurious structure due to discretization. This includes step size in α and β, as well as step size in sin(b) and γ. However, the trends and nature of the situation are clear. The one-D histogram shows that the observational density is not uniform over the celestial sphere, but the under-observed portions have more than 70% the average observation density. I do not find this troublesome. The peak is at $\pi/2 - \xi$, which is not surprising.

In the two-D histograms, there is a curved edge to the region that contains observations. This takes the form given by Eq 6, and should be sharp. (The observation density is proportional to $1/\sqrt{\tilde{\gamma} - |\gamma|}$ for $|\gamma| < \tilde{\gamma}$ near the edge and zero for $|\gamma| > \tilde{\gamma}$.) However, the histogram binning flattens the spike near the edge, and the histogram interpolation introduces a slope where a precipitous drop should be. The maximum value of γ for which there are observations at all latitudes is ξ. The minimum latitude at which γ takes all values is $\pi/2 - \xi$. These results are easily confirmed by inspection of Figs 2, 3, and 4.

For $\xi = 45$ deg, we consider three cases: (1) at the ecliptic (b = 0), half of the observations are for 30 deg $< |\gamma| <$ 45 deg; (2) at b = 30 deg (which divides the area of the hemisphere evenly, sin(b) = 0.5), half of the observations are for 35 deg $< |\gamma| <$ 55 deg; and (3) (well) above b = 45 deg, the distribution is nearly uniform in γ. In no case is there likely to be much elongation of the error ellipse due to the observing geometry. A more difficult question, which will not be addressed here, is: What is the coverage in γ for real observations, for which the averaging over ℓ does not take place? An important factor in the answer to this question is likely to be the number of precession cycles per year (or if they are well interleaved, per mission). This consideration favors faster precession, and thus faster spin, but the question must be addressed quantitatively.

IV. Acknowledgments

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V. Distribution

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