Likelihood Centroiding of CCD Point Spread Functions

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Abstract

Traditionally, least squares has been used for fitting point spread functions (PSFs) to observed CCD data. However, and particularly at low signal levels, the ability of CCDs to count individual photons and the CCD readout noise conspire to produce a likelihood function that may differ significantly from the standard least-squares likelihood. This suggests that modeling the likelihood function directly may result in improved centroiding. In this paper I report on some preliminary investigations relevant to the FAME project on likelihood fitting of PSFs.

Assumptions

- The PSF f(x) is assumed one dimensional and known perfectly (for this simulation assumed Gaussian).
  - For FAME the one-dimensional approximation is reasonable since for practical purposes the y-direction is binned by the hardware anyway
  - Perfect knowledge of PSF is unrealistic but allows us to separate the effects to be studied from other effects.
  - Gaussian assumption not critical since we are not investigating the dependence on the exact shape of the PSF.
  - Parameters chosen to approximate the width of FAME PSF
- Photons are detected as Poisson events with probability obtained by integrating the PSF over the pixel

Complications

- For FAME, two complications need also to be considered.
  - Readout noise is significant at ~7 electrons, and is Gaussian (Jim Phillips, private communication)
  - The A/D converters result in several different electron counts being digitized in the same A/D output level—about 2.5 electrons per level at the lowest scale (Jim Phillips, private communication)
- I also calculated the case for no readout noise and no A/D quantization effects

No noise, no A/D stepping

- Let a be the expected photon count in a pixel. Then the log likelihood function given n detected electrons is

$$\ln\left(\frac{a^n e^{-a}}{n!}\right) \approx n\ln a - a - n\ln n - n \text{ (using Stirling's approximation)}$$

$$= n\ln a + \Delta a - n\ln(a+\Delta a) \text{ (with } \Delta a = n - a)$$

$$= n\ln a + \Delta a - n\ln a - n\ln(1+\Delta a/a)$$

$$= \Delta a + (a+\Delta a)(\Delta a/a - \Delta a^2/2a^2 + \ldots) \text{ (expanding log)}$$

$$= -\frac{1}{2}\frac{\Delta a^2}{a} + \ldots$$

- For large a, this is equivalent to the least-squares log likelihood function with weights $1/a$. This shows that maximum likelihood Poisson estimation recovers least-squares for large a.

Results From Simple Model

- Investigate the standard deviation σ of centroid using exact likelihood versus the σ from least-squares approximation, for various expected number of counts (30-10000, by factors of half a decade). The results show a modest penalty of up to about 13% in the standard deviation if the least squares likelihood function is used instead of the correct likelihood function.

Expected Counts	Centroid σ (Pixels) Exact	Least Squares	Penalty (%)
30	0.206	0.233	12.83
100	0.108	0.117	8.75
300	0.061	0.067	8.98
1000	0.033	0.036	9.08
3000	0.019	0.020	7.31
10000	0.011	0.011	0.77

Readout Noise and A/D Binning

- The observed datum is d, the particular quantization level observed from the A/D converter.
- The derivation of the likelihood function is rather involved. The result is

$$p(d \mid a) = e^{-a} \sum_{k=0}^{\infty} \frac{a^k}{k!} \left[\Phi\left(\frac{d - b + 1 - gk}{g\sigma} \right) - \Phi\left(\frac{d - b - gk}{g\sigma} \right) \right] \text{ where}$$

$a =$ expected photon count

$d =$ observed quantization level

$b =$ zero-point of quantization

$g =$ gain (quantization of one photon for A/D)

$\sigma =$ readout noise standard deviation, in electrons

$\Phi =$ cumulative standard normal distribution function

Results

- I calculated the same quantities as for the simple case with several choices of the parameters: g =0 and 0.4 (the latter corresponding the the FAME A/D converters in their current incarnation), σ = 0 and 5 electrons, and b =0.
- Interestingly, the results for the exact treatment are only slightly better than least squares, except at the very lowest number of counts (30, corresponding to magnitude 18.5 for FAME). Here however, the improvement can be over 100%.
- Thus, the loss of single-photon information that occurs due to the “smearing-out” effects of readout noise and the binning together of several photons in one quantization level by the A/D converter makes least squares competitive except for the faintest stars.

Tabular results

Condition	Expected Counts	Centroid σ (Pixels) Exact	Least Squares	Penalty (%)
Gain 0.4	30	0.231	0.500	115.97
	100	0.112	0.115	2.56
	300	0.061	0.062	2.02
	1000	0.033	0.033	0.53
	3000	0.019	0.019	0.29
	10000	0.010	0.010	0.00
RO Noise 5 electrons	30	0.724	0.998	37.79
	100	0.194	0.195	0.24
	300	0.084	0.084	0.11
	1000	0.038	0.038	0.06
	3000	0.021	0.021	0.05
	10000	0.011	0.011	0.07
RO Noise 5 electrons *and* Gain 0.4	30	0.761	1.081	41.98
	100	0.194	0.195	0.59
	300	0.084	0.084	0.12
	1000	0.038	0.038	-0.12
	3000	0.020	0.020	-0.01
	10000	0.011	0.011	0.00

Conclusions

- If readout noise is very low (probably not more than 1-2 electrons) and data are quantized by the A/D converter so that 1 count corresponds to 1 quantization level, then modest improvements in the standard deviation of 10-15% can be achieved by maximizing the exact likelihood, for signal levels up to something under 10,000 counts.
- With the readout noise and quantization levels currently contemplated with FAME, significant improvements are attained only at the lowest counting level; however, the improvements are significant, from ~40% to well over 100%; and most stars are faint.

Speculations

- There was not time to investigate the situation where we take the zero-point $b < 0$, which could be accomplished by suitable hardware design.
- I conjecture that some of the shortfall at the low end may be due to the failure to quantize signals less than zero in this simulation (i.e., designing the hardware so that $b < 0$ may improve things for least squares).