A Formalism for Computing FAME Observational Profiles

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1 Introduction

This note is about computing the shape of the 1D or 2D star images that FAME collects and sends to the ground, one per observation. It is an attempt to represent in mathematical notation all of the non-stochastic effects that contribute to the observed image. This is important if we wish to construct a template function for centroiding that contains as much *a priori* information as possible. In particular, it would seem that such a computation would be necessary if we wish to connect the template function to the position of the observed star on the sky. Using a tailored template function for each observation may not be the only way to successfully perform the centroiding operation for FAME, but analysis by Makarov [1] indicates that this scheme has the potential to virtually eliminate centroiding bias.

In this note I make a distinction between the *observed image* and the *observational profile*. As used here, the observed image is the ensemble of pixel intensities as received for a particular observation. The observational profile is the underlying shape of the of the image in the oversampled and noise-free case — essentially, the parent function from which the observation is drawn.

2 Observation Description

FAME will have 13 CCDs in its focal plane, each of which contains 2048×4096 pixels. As FAME rotates, the star images move across the focal plane. The direction of image movement in the focal plane is referred to as the in-scan direction, and is described by the coordinate u; the orthogonal direction is called cross-scan, and is described by the coordinate v. The (u, v) origin is at the center of the focal plane and u increases in the direction of star image motion. The CCDs operate in TDI mode, with the charge following the star images as they move across the chips. All of the CCDs are oriented with their long dimensions in the in-scan direction. It takes 2.24 seconds for a star image to move across one CCD, and a star image may cross several CCDs in succession. The CCD pixel size, 15 μm, projects onto the sky as 0.295 arcsec. The core of the point-spread function (PSF) of the instrument is approximately 1.5 pixels FWHM in the in-scan direction and 5 pixels wide in the cross-scan direction. The exact PSF depends on the star's spectral type and the position of the star image in the focal plane.

Each observation represents the charge accumulated on a small rectangular area of one CCD, 13 pixels in-scan by 24 pixels cross-scan, that is centered on the point-spread-function (PSF) of the star observed. This observational "postage stamp" can be thought of as a group of virtual pixels that moves with the star across the CCD; at different times, different groups of physical pixels are involved. Because the postage stamp is co-moving with the star, the 2.24-second crossing time of the star image across the CCD is the integration time of the observation.

There are two important aspects of the TDI clocking that affect the final image shape. First, although the star image moves smoothly across the CCD, the charge can only be moved in discreet one-pixel steps. This introduces a one-pixel smearing into each image in the in-scan direction, essentially the PSF convolved with a one-pixel-wide boxcar function. Secondly, the TDI rate is not, in general, matched exactly to the star image motion rate. So, as the star image moves across the CCD, the charge concentration near the center of the postage stamp gradually shifts either behind or ahead of the star image, causing further smearing. This "TDI mismatch" is supposed to be limited to not more than about one pixel during one 2.24-second integration time. In attempting to predict an observation profile, it is clear that we need to distinguish between the photon centroid (the PSF) and the charge centroid and to separately account for their positions on the focal plane as a function of time.

Furthermore, the PSF is itself problematic; we actually have to deal with a large family of PSF functions. The PSF for a particular star is essentially the sum of a series of monochromatic PSFs, weighted by the received stellar photon flux as a function of wavelength. The width of each monochromatic PSF is directly proportional to its wavelength, but the exact shape depends on the position in the focal plane. Furthermore, because FAME's optical design has significant lateral color separation (for our application), the peaks of the monochromic PSFs are not coincident and these offsets too are a function of position on the focal plane. Therefore, apart from the TDI considerations, the charge moving across a CCD reflects the accumulation of electrons from a continuously changing parent distribution (or set of parent distributions).

People sometimes use the phrase "observed PSFs". For the case of FAME, this is a misuse of the term PSF and can lead to sloppy thinking. FAME does not observe the instrumental PSF; the observed image is the product of many processes acting on an entire ensemble of PSFs.

Since all of these effects depend critically on the star image position in the focal plane as a function of time, that position must be continuously — and accurately — modeled. FAME's rotation is complex on small angular scales and includes both precession and nutation components. The precession imparts a significant cross-scan motion (up to 4 pixels per crossing time) to the star images. At the precision with which FAME must operate, higher-order rotational motions may be important as well, even over time scales as short as several seconds. Assuming a linear track of the star image across a CCD may not be sufficient for computing an accurate observational profile.

3 Definitions

We must first decide what we mean by the position of a star in the focal plane at some instant. This is not a trivial exercise since we could not observe the star as a point even if FAME had no rotation and the CCDs were operated in "stare" mode. What we would observe in that case is a distribution of photons that is not unique for the star's position on the sky; if the star's color were to change, the center of the distribution would also change. Of course, for our definition we could pick a specific wavelength near the center of the observing band, say 700 nm. But 700 nm is unlikely to be either the peak or average wavelength of the photon flux of a particular star. Furthermore, choosing one special wavelength would mean that all color-dependent effects would then be measured relative to our choice.

Therefore I propose that the *geometric position* of a star in the focal plane, $\mathbf{p}_\star = (u_\star, v_\star)$, be defined as the two-dimensional position (at some specific time) of the apex of the converging bundle of rays from the star, assuming perfect geometric optics. In this geometric definition the optical system is assumed to have no distortion and no color separation. "No distortion" means that the mapping from the celestial sphere onto the focal plane has a constant scale, so that equal arcs on the sky anywhere in the field of view map to equal linear separations in the focal plane. (This has been the design goal for the FAME optics, although it is *not* the more common gnomic projection.) In essence, the geometric position of the star is is an imaginary point in an imaginary optical system — the only attribute it shares with the real system is the focal length — but it provides a point of reference for the rest of the development. This point can be easily related to a position on the celestial sphere.

Suppose $\mathbf{p}_p(\mathbf{p}_\star, \lambda) = (u_p, v_p)$ is the focal plane position of the peak (maximum) of the monochromatic PSF, at wavelength λ, of the star at geometric position $\mathbf{p}_\star$. (We assume the star is a point source.) Then the distortion of the optical system can be operationally defined as the difference between these two positions. The distortion $\mathbf{D}$ is a vector function of wavelength and position on the focal plane:

$$\mathbf{D}(u, v, \lambda) = \mathbf{p}_p(\mathbf{p}_\star, \lambda) - \mathbf{p}_\star = \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = \begin{pmatrix} u_p \\ v_p \end{pmatrix} - \begin{pmatrix} u_\star \\ v_\star \end{pmatrix} \tag{1}$$

We assume that $|\mathbf{D}|$ is everywhere quite small compared to the focal plane dimensions and that the variation of $\mathbf{D}$ is negligible over lengths of order $|\mathbf{D}|$; therefore it is not actually necessary to distinguish whether u and v in $\mathbf{D}(u, v, \lambda)$ refer to the position of the PSF or the geometric position of the star. For specificity, we will put $u = u_\star$ and $v = v_\star$. This definition of distortion contains the lateral color separation of the instrument, so a separate accounting of that does not have to be made. Although this definition uses the peak of the PSF as the basic reference point, some other well-defined point within the PSF — such as the first moment — could be used instead, as long as the functional form of the PSF could be developed with respect to the chosen point. In fact, the optical-design software ZEMAX produces computed PSF functions with respect to a 2D coordinate system that is centered at the geometric position of the star. In such a case, $\mathbf{D}$ effectively disappears as a separate entity.

The object of the game is to estimate the counts of electrons in the observational postage stamp due to received photons (this note does not concern itself with electronic noise). There are two separate problems: the first is to compute the counts on a per-pixel basis, that is, to predict what the actual observational image would be. But the FAME image is somewhat undersampled in the in-scan direction, that is, the PSF extends over only a small number of pixels. Therefore, the distribution of counts in the image is very sensitive to small changes in the star's focal plane position, due to bad *a priori* astrometry or incorrect modeling of any number of effects. So if we want to compute what I call the observational profile — the shape of the image in the oversampled and noise-free limit — we need to divide the postage stamp into much smaller computational units called *subpixels*. For simplicity, I assume that the subpixel width is just the pixel width divided by an integer, k, that we can set. To compute the profile, then, the postage stamp becomes a much finer mosaic of infinitesimal areas, made up of a large number of subpixels: k^2 times the number of pixels (the latter is usually 13×24). Computing the counts in the actual observation then can be considered to be a special case of subpixelation where $k = 1$ so that the subpixels are pixels. Of course, we can always compute pixel counts after the fact regardless of the size of the subpixels.

With these basic definitions in hand, we can go on to describe other quantities and functions. I assume that all the pixels on a CCD have the same sensitivity, that the sensitivity is uniform within a pixel, and that there is no "dead space" between pixels or near their edge. I also assume that in TDI mode, at each row-shift epoch, the entire charge in one row of pixels is transferred instantaneously, without noise or loss, to the next row and entirely replaces the previous charge in that row. At some level these assumptions are not true, of course, but once the basic development is presented it is easier to see how more realistic models can be incorporated, and some comments on that will be presented in Section 7. Mention is made below of the "lower left corner" of various entities on the focal plane. This refers to the focal plane as usually drawn, with u increasing to the right and v increasing upward. The lower left corner is then the corner with minimum u and minimum v.

$\mathbf{p}_\star(t) = (u_\star(t), v_\star(t))$	Track of geometric position of the star across the focal plane.
$P(x, y; u, v, \lambda)$	Normalized point-spread function at wavelength λ with maximum at (u, v) in the focal plane. x and y are measured in the u and v directions, respectively, from the maximum. P encompasses a family of surfaces in (x, y), one at each point in (u, v, λ) parameter space. Units are area^{-1}. Normalization means that at a given (u, v, λ), the volume under the $P(x, y)$ surface is 1.
$T(\lambda)$	Normalized throughput of instrument system at wavelength λ, including factors from CCD quantum efficiency, filters, light loss due to optical coatings and materials, and deposition. We assume here that T is a constant for any one CCD. Normalization means that the area under the curve is 1.
$S(\lambda)$	Stellar spectrum — the natural photon flux from a star, as a function of wavelength, through one of the instrument apertures (40×9 cm^2). Expressed as number of photons/second/unit wavelength.
W	Width of a pixel (assumed square) in units of u or v.
w	Width of a subpixel (assumed square) in units of u or v, equal to W divided by an integer k.
k	Number of subpixel widths in a pixel width, an integer. The number of subpixels in a pixel is k^2. A special case is W=w and k=1, where the subpixels and pixels are the same.
n, m	Dimension of the postage stamp expressed as number of subpixels in u and v directions, respectively. The normal size of the postage stamp is $n = 13k$ and $m = 24k$.
i, j	Subpixel index within the postage stamp, integers. $0 \leq i \leq n - 1$ and $0 \leq j \leq m - 1$. Subpixel (0,0) is at the lower left corner of the postage stamp.

u', v'	Focal-plane coordinates within the postage stamp with respect to its lower left corner. u' and v' are real and non-negative and run from (0,0) to (nw,mw). Pixel borders are at values of u' and v' that are integer multiples of W; subpixel borders are at integer multiples of w. Note that since the postage stamp is shifted across the CCD at the TDI rate, i, j, u', and v' can be considered to be in a moving reference frame.
u_0, v_0	u and v coordinates of the lower left corner of the photosensitive area of the CCD, specifically, the lower left corner of CCD pixel (0,0).
t_0	Time at which the leading (right) row of pixels in the postage stamp first appears at physical row 0 of CCD.
c_0	Physical column number of CCD corresponding to lowest column of pixels in the postage stamp.
TDI	TDI rate expressed as number of pixel rows shifted per second. Note that this is not affected by subpixelation; the charge always shifts in units of pixels.
τ	Time between TDI row-shifts. $\tau = 1/\mathrm{TDI}$. Assumed to be the exposure time of the pixels (or subpixels) in the postage stamp at each TDI step.
t_r	A row-shift epoch, the instant at which a row shift is completed. t_0 is the first row shift epoch relevant to a given postage stamp. Succeeding t_r's are at $t_0 + \tau,\ t_0 + 2\tau,\ t_0 + 3\tau,\ \ldots$

4 Basic Relations

Using the above definitions, we find the following relations:

Row number on CCD of right (leading) row of postage stamp	$\mathrm{TDI} \times (t_r - t_0)$

u coordinate of right edge of postage stamp	$u_0 + [\mathrm{TDI} \times (t_r - t_0) + 1]k\mathrm{w}$
u coordinate of left edge of postage stamp ($u' = 0$)	$u_0 + [\mathrm{TDI} \times (t_r - t_0) + 1]k\mathrm{w} - n\mathrm{w}$
v coordinate of lower edge of postage stamp ($v' = 0$)	$v_0 + c_0\mathrm{W}$
Position of peak of monochromatic PSF at wavelength λ on focal plane, corresponding to star at geometric position $\mathbf{p}_\star$.	$\mathbf{p}_p(\mathbf{p}_\star, \lambda) = \mathbf{p}_\star + \mathbf{D}(\mathbf{p}_\star, \lambda)$
u' coordinate of monochromatic PSF (at wavelength λ)	$u'_p = u_\star + \delta u - u_0 - [\mathrm{TDI} \times (t_r - t_0) + 1]k\mathrm{w} + \mathrm{nw}$ where δu is the u-component of $\mathbf{D}(u_\star, v_\star, \lambda)$
v' coordinate of monochromatic PSF (at wavelength λ)	$v'_p = v_\star + \delta v - v_0 - c_0\mathrm{W}$ where δv is the v-component of $\mathbf{D}(u_\star, v_\star, \lambda)$

5 Formulas

We now come to the computation of the observational profile, the estimate of the intensity of every subpixel in the postage stamp. It is advantageous to develop this computation in terms of the moving postage stamp coordinates, so that individual postage stamp subpixels retain their identity throughout. This means that the position of the star's monochromatic PSFs, which move with the star across the focal plane, have to be repeatedly transformed to the postage stamp coordinate system. Note that the PSFs are not stationary in this system, since the PSFs move continuously whereas the postage stamp moves only at row-shift epochs. Furthermore, due to TDI mismatch, even the average rate of motion of the PSFs and the postage stamp may not be the same.

At an arbitrary point with postage stamp coordinates (u', v'), we first want to compute the intensity of the PSF due to a star at geometric position $\mathbf{p}_\star(t)$. The monochromatic PSF of the star at wavelength λ will have its peak at $\mathbf{p}_p(\mathbf{p}_\star(t), \lambda) = \mathbf{p}_\star(t) + \mathbf{D}(\mathbf{p}_\star(t), \lambda)$. We will designate the postage stamp coordinates of the PSF peak simply by (u'_p, v'_p), the expressions for which are given in the previous section. The value of the PSF at the arbitrary point will then be

$$P(u' - u'_p, v' - v'_p; u_\star, v_\star, \lambda) \tag{2}$$

where we have dropped the functional dependencies for now.

The received flux of photons through an infinitesimal area $\mathrm{d}u'\,\mathrm{d}v'$ at this point, during a time interval $\mathrm{d}t$ and within a wavelength interval $\mathrm{d}\lambda$ centered on λ is

$$S(\lambda)\,T(\lambda)\;P(u' - u'_p, v' - v'_p; u_\star, v_\star, \lambda)\;\mathrm{d}u'\,\mathrm{d}v'\,\mathrm{d}\lambda\,\mathrm{d}t \tag{3}$$

The received flux of photons over the entire subpixel (i, j) that contains the point (u', v'), and over the entire bandwidth of the instrument is then

$$\int_{\lambda_1}^{\lambda_2} S(\lambda)\,T(\lambda) \int_{j\mathrm{W}}^{(j+1)\mathrm{W}} \int_{i\mathrm{W}}^{(i+1)\mathrm{W}} P(u' - u'_p, v' - v'_p; u_\star, v_\star, \lambda)\,\mathrm{d}u'\,\mathrm{d}v'\,\mathrm{d}\lambda\,\mathrm{d}t \tag{4}$$

where $\lambda_1 \approx 500\,\mathrm{nm}$ and $\lambda_2 \approx 900\,\mathrm{nm}$. The w's in the integration limits represent the width of the subpixels. This expression must be integrated over time. The postage stamp is stationary on the CCD between row shifts. The star's monochromatic PSFs are moving continuously; however, expressed in postage stamp coordinates, their movement is continuous only over the row-shift interval τ. The interval begins at row-shift epoch t_r.

$$\int_{t_r}^{t_r+\tau} \int_{\lambda_1}^{\lambda_2} S(\lambda)\,T(\lambda) \int_{j\mathrm{W}}^{(j+1)\mathrm{W}} \int_{i\mathrm{W}}^{(i+1)\mathrm{W}} P(u' - u'_p, v' - v'_p; u_\star, v_\star, \lambda)\,\mathrm{d}u'\,\mathrm{d}v'\,\mathrm{d}\lambda\,\mathrm{d}t \tag{5}$$

This expression represents the contribution to the electron count for subpixel (i, j) over one short integration interval, the time between row shifts ($\sim$0.5 ms). For an observation where the pixels never saturate (at $\sim 10^5\,e^-$) the result should be a number $< 25/k^2$, which should be stored as a floating-point value. For an actual observation the integer count of electrons per subpixel per row-shift interval would come from a Poisson distribution with the above result as the mean, but for purposes of obtaining the (noiseless) observational profile this detail is not relevant.

We must add up the contributions over all 4096 row shifts. In doing so, we finally obtain the intensity $\mathcal{I}$ (electron count) for subpixel (i, j) in the absence of any type of noise:

$$\mathcal{I}(i, j) = \sum_{L=0}^{4095} \int_{t_r}^{t_r+\tau} \int_{\lambda_1}^{\lambda_2} S(\lambda)\,T(\lambda) \int_{j\mathrm{W}}^{(j+1)\mathrm{W}} \int_{i\mathrm{W}}^{(i+1)\mathrm{W}} P(u' - u'_p, v' - v'_p; u_\star, v_\star, \lambda)\,\mathrm{d}u'\,\mathrm{d}v'\,\mathrm{d}\lambda\,\mathrm{d}t \tag{6}$$

where L is the count of row shifts, $t_r = t_0 + (L + n/k - 1 - \mathrm{int}[i/k]\,)\tau$, and int[ ] is the largest-integer function. The expression for t_r accounts for the fact that different sets of subpixels start their exposure at different times; for example, only subpixels that are in the right (leading) row of pixels in the postage stamp begin exposure at time t_0.

We use eqn. (6) to compute the intensities of all the subpixels in the postage stamp. If $k \geq 2$ this is the 2D observational profile. If $k = 1$ then the subpixels are pixels and we have directly obtained an estimate of the 2D observed image. We can always obtain the latter from the former simply by summing up all the subpixel intensities within each pixel. For example, suppose we want $\mathcal{I}'(I, J)$, the computed intensity of pixel (I, J) in the postage stamp:

$$\mathcal{I}'(I, J) = \sum_{j=0}^{k-1} \sum_{i=0}^{k-1} \mathcal{I}\,(Ik + i, Jk + j) \tag{7}$$

That is the 2D image. The binned 1D image is just as easily obtained:

$$\mathcal{I}''(I) = \sum_{j=0}^{m-1} \sum_{i=0}^{k-1} \mathcal{I}\,(Ik + i, j) \tag{8}$$

6 Computational Strategy

The PSF integrand $P(\ldots)$ in eqn. (6) appears relatively harmless but that is only because we have not shown the functional dependencies of its parameters. In fact, if all the functional dependencies in P are explicitly displayed, we have

$$\begin{aligned} P\ (\ & u' - [\ u_\star(t) + \delta u(u_\star(t), v_\star(t), \lambda) - u_0 - \{\mathrm{TDI}\,(t_r - t_0) + 1\}k\mathrm{w} + n\mathrm{w}\ ], \\ & v' - [\ v_\star(t) + \delta v(u_\star(t), v_\star(t), \lambda) - v_0 - c_0\mathrm{W}\ ]; \\ & u_\star(t),\ v_\star(t),\ \lambda\ ) \end{aligned} \tag{9}$$

which gives a better perspective on the computational task.

Clearly at each integration point the first task is to obtain the geometric position of the star. The data analysis pipeline as designed has the capability to compute the focal plane coordinates of any star at any time, but the models it uses for this would probably not be accurate enough for the observational profile calculation until a spiral reduction is completed for the rotation containing the observation. The star motion across the focal plane is nonlinear at a level (mas) that may matter for this computation but it should be possible to represent the track as a low-order polynomial in time (unless jitter is significant). If we leave this task to "other parts" of the pipeline then the rapid computation of $(u_\star(t), v_\star(t))$ for arbitrary times t in the above integral evaluation is not an issue.

The distortion function $\mathbf{D}(u\star(t), v_\star(t), \lambda)$ is more of an evaluation problem. It will probably have to be interpolated from a set of stored monochromatic distortion values for a grid of focal plane positions, although a more convenient analytic representation may be possible. However, the distortion does not have to be evaluated very often, because its maximum gradient for the current (October 2001) optical design seems to be less than 10^{-3} (microns of distortion / microns of focal plane position). That means that over one row-shift interval — during which the star moves approximately one pixel — we can use a single distortion vector per wavelength, with an error that is a fraction of the final centroiding accuracy requirement.

This brings us to the evaluation of the point-spread function. This requires dealing with a library of empirically-determined PSFs across the focal plane and at various wavelengths. Rather than attempting interpolation within such a huge number of data points, Makarov [2] shows that the PSFs can be compactly represented by a relatively low-order fit to Hermite orthogonal functions. Still, the numerical load involved in eqn. (6) depends critically on how many times the PSF-evaluation routine has to be interrogated. Clearly, eqn. (6) seems to imply a very large number of such operations. To be more quantitative about it, we have to know how accurate the integrations need to be — information that depends on the required accuracy of the result.

Valeri's tech note [1] concluded that to eliminate centroiding biases the template function must match the actual observational profile to 0.5% for FAME's 550–850 nm bandpass. This would be the accuracy with which each of the subpixel intensities $\mathcal{I}(i,j)$ should be computed using eqn. (6). Assuming we know the on-orbit PSFs to infinite precision (somewhat of a stretch!) how good do the integrals have to be? Well, the subpixel intensities $\mathcal{I}(i,j)$ involve a summation over 4096 row-shifts. Since the star image stays near the center of the postage stamp the entire time, it is not too much of an approximation to say that the final computed intensity of any given subpixel is the result of the sum of 4096 nearly-equal contributions. If the numerical error in each of these contributions is uncorrelated and noise-like, then the fractional error of each needs to be not worse than $\sqrt{4096} \times$ the required fractional error of the result, i.e., $64 \times 0.5\% = 32\%$. [1]

That's pretty good news, and leads us to hope that the integrals in eqn. (6) can be evaluated with simple summations and that sophisticated numerical methods will not be required. In playing around with the sinc function and its first two derivatives, it seemed to me that we could do quite well without actually doing the surface integral ($\int\int \cdots \mathrm{d}u'\mathrm{d}v'$) at all if the subpixels were small enough: simply multiply the value of P for the center of the subpixel by the subpixel area (w^2). The result is exact in the case where the $P(u', v')$ surface has no curvature over the subpixel. That

[1] Beware of possible swindle here... check logic carefully! Note that, in any case, the 32% tolerance does not apply to errors that would systematically affect all the contributions, such as an incorrectly computed star track.

is not the case but the error for this part of the calculation will certainly be less than 32% (except near the zeros of P, which are not important for centroiding) as long as $k \geq \sim 4$. We probably need $k \geq 4$ anyway just to get adequate spatial sampling of the observation profile.

That leaves us with the two outer integrals in eqn. (6). Several people have been constructing polychromatic PSFs by adding together, for each one, about ten monochromatic PSFs. Based on this work I will propose a simple summation over nine 50-nm wavelength intervals as adequate without attempting any further analysis. The outermost integral is over time and accounts for the changing position of the PSF during one row-shift interval. The star's PSF moves about one pixel per row-shift interval τ, or one subpixel in τ/k. Since we have accepted one subpixel as an adequate differential area (as long as $k \geq 4$) for sampling the PSF function P to the required accuracy, it is reasonable to set τ/k as the differential time interval for the outermost integral.

So the form of eqn. (6) for practical evaluation becomes

$$\mathcal{I}(i,j) = \sum_{L=0}^{4095} \sum_{\kappa=1}^{k} \sum_{l=1}^{9} S(\lambda_l)\, T(\lambda_l)\, P\,(\,(i+\frac{1}{2})\mathrm{w} - u_p',\, (j+\frac{1}{2})\mathrm{w} - v_p';\, u_\star(t_\kappa),\, v_\star(t_\kappa),\, \lambda_l\,)\, \mathrm{w}^2\, \Delta\lambda\, \Delta t \tag{10}$$

$$\begin{aligned}
\text{where} \quad t_r &= t_0 + (\,L + n/k - 1 - \mathrm{int}[i/k]\,)\,\tau \\
t_\kappa &= t_r + (\,(\kappa - \frac{1}{2})/k\,)\,\tau \\
\lambda_l &= 50l + 450 \\
\Delta\lambda &= 50 \\
\Delta t &= \tau/k \\
u_p' &= u_\star(t_\kappa) + \delta u(u_\star(t_1), v_\star(t_1), \lambda_l) - u_0 - \{\mathrm{TDI}\,(t_r - t_0) + 1\}k\mathrm{w} + n\mathrm{w} \\
v_p' &= v_\star(t_\kappa) + \delta v(u_\star(t_1), v_\star(t_1), \lambda_l) - v_0 - c_0\mathrm{W}
\end{aligned}$$

and where λ_l and $\Delta\lambda$ are expressed in nanometers (nm),

the distortion offsets δu and δv are evaluated only once per row-shift interval (each L) for each wavelength λ_l, using the star position at time t_1, and

$u_\star(t)$ and $v_\star(t)$ are low-order polynomials in t.

7 Concluding Remarks

The number of times the function P has to be evaluated in eqn. (10) is $4096 \times k \times 9 = 36{,}864k$. But to obtain the entire observational profile $\mathcal{I}(i,j)$ has to be evaluated $13k \times 24k$ times. So the final number of evaluations of P for the entire postage stamp is $1.15 \times 10^7\,k^3$. Needless to say, that's a lot of CPU cycles, even if P can be efficiently evaluated.

All kinds of computational shortcuts suggest themselves. For example, since P is much wider in the cross-scan than in-scan direction, we could create subpixels that were much taller than wide. We would end up with two values for w (w_u and w_v) and two values for k (k_u and k_v), for the in-scan and cross-scan directions, respectively. Since $k_v \approx k_u/4$, it is possible that we could get away with $k_v = 1$. Generalizing eqn. (10) in such a way is quite simple.

Furthermore, there is no requirement that the value of k be the same for all the virtual pixels in the postage stamp. The maximum value of k (whatever value is chosen) is needed only near the center of the PSF. Lower spatial resolutions would suffice for the wings. So a factor of two or three in computational speed for the entire postage stamp could be obtained just by making k a function of i and j. Such a strategy would not change eqn. (10).

The summation over wavelength intervals also deserves a look. It is possible that there might be no significant loss of precision in reducing the number of wavelength intervals. Or, instead of storing nine monochromatic PSF functions for each area on the focal plane, and an equal number of monochromatic distortion vectors, perhaps it would be more efficient to store a set of pre-computed polychromatic PSFs for each area, one for each of 10 or so values of a star's effective surface temperature (T_{eff}). Such a scheme would turn the summation over wavelength intervals into an interpolation along the T_{eff} axis. This kind of interpolation (if efficiently implemented) would likely involve fewer evaluations of P than the summation, albeit at a small cost in additional overhead.

Of course, the only way to reliably assess any option for evaluating eqn. (10) is to try it. We assume that the results of eqn. (10) asymptotically approach "truth" (within the limits of the crude model of the CCD on which it is based) as k increases. The results of any computational shortcuts — including successively smaller values of k — should be evaluated against the large-k case.

On the other hand, it might be necessary to make eqn. (10) more complicated, to incorporate a more realistic CCD model. For example, if the CCD pixels do not all have the same sensitivity, then we would have to keep track of which CCD pixel includes subpixel (i,j) of the postage stamp during each row-shift interval, and apply the appropriate sensitivity factor. Other possible extensions include taking account of the border of reduced sensitivity around each CCD pixel, and a three-phase charge transfer cycle at each row shift.

Finally, could eqn. (10), or something comparable, be used for actual data analysis? Unfortunately, it appears that this would be several orders of magnitude too slow. On my 1.5 GHz PC, evaluation of a simplified version of eqn. (10) using a 2D single-precision sinc^2 function as the PSF takes about a half second, and an entire 13×24 postage stamp with $k=4$ takes about 30 minutes! Admittedly, this is without any of the above-mentioned shortcuts, and the executable was generated by an old 16-bit Fortran compiler. Still, coding eqn. (10) in C on a fast machine, with all possible computational shortcuts implemented, is unlikely to reduce the total evaluation time per postage stamp below a minute or so. We will be receiving several hundred observations per second! (This could also be a problem for the data simulator, which will have to incorporate some process analogous to eqn. (10) to generate realistic observed images.) The usefulness of eqn. (10) will probably be in providing "truth" data to compare to more efficient means of generating template functions for centroiding. Or, if it turns out that the observational profiles are relatively insensitive to things like star-path curvature and CCD column number, it might make sense to use eqn. (10) to pre-compute a large library of observational profiles that could be rapidly interpolated during data analysis.

References

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