Computing Partials for Astrometric Parameters; Computing Ecliptic Coordinates

Addenda to "Algorithms for Rapidly Computing the FAME Observing Sequence"

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1 Introduction

This note provides formulas for two topics not covered in [1], the primary purpose of which was to describe a method for efficiently computing the observing sequence by FAME's on-board computer system. Since the formulas given in that technical note have also been used for the prototype data analysis pipeline, it seems appropriate to add two topics not originally covered. These two topics, on computing the partials for astrometric parameters and on converting to ecliptic coordinates, would only be applicable to the ground-based data analysis, not the on-board system.

2 Partial Derivatives for Astrometric Parameters

2.1 Overview of Parameter Estimation

The ultimate goal of the FAME astrometric data analysis pipeline is the formation of a least-squares solution which yields, for each star, the values of five astrometric quantities: position offset in RA and Dec, proper motion in RA and Dec, and parallax. Actually, for many stars, proper motion may be generalized to a time series to account for binary orbital motion. These parameters must be determined along with a large number of other parameters that describe the spacecraft's instrument and attitude history. The formation of the least squares solution is based on the computation, for each observation, of partial derivatives of the observed quantities with respect to all these parameters. In its most general form, the conditional equation for a single

FAME observable for a single observation is thus

$$(O-C)_u = \sum_i \frac{\partial u}{\partial p_i}\Delta p_i + \sum_j \frac{\partial u}{\partial q_j}\Delta q_i + \sum_k \frac{\partial u}{\partial r_k}\Delta r_i \tag{1}$$

where u is an observable quantity (e.g., the star image transit time across a CCD row), $(O-C)_u$ is the difference between the observed and computed values for u, and Δp_i, Δq_i, and Δr_i are corrections to the astrometric, attitude, and instrument parameters, respectively. The partial derivatives for a given observation become elements in a row of the solution design matrix. In practice, separate solutions may be used for the three classes of parameters. See [2] for a more complete description of the solution strategy.

In this note we are interested in only the partial derivatives for the astrometric parameters. Makarov [3] has already developed formulas for the astrometric partials based on a rotation matrix development. The following development is based on the vector development given in [1] and may be conceptually and computationally simpler if one already has implemented the algorithms in that note and the vectors referred to below are therefore immediately available. The two approaches seem to be equivalent.

2.2 Partials for Position and Proper Motion

In Section 2.3 of [1], a coordinate system was established on the instrument focal plane with basis vectors $\mathbf{u}$ and $\mathbf{v}$ and origin at the point where the optical axis intersects the focal plane. We can imagine this coordinate system projected onto the sky at the point where the extension of the optical axis (the aperture boresight) meets the celestial sphere. The $\mathbf{u}$ vector then points nearly opposite the instantaneous direction of FAME's scanning motion. Therefore, the [$\mathbf{u}$,$\mathbf{v}$] system takes on a completely arbitrary and changing direction with respect to the ICRS axes. However, the FAME observables are most simply represented in this system since the CCDs are (nominally) aligned to the [$\mathbf{u}$,$\mathbf{v}$] axes. In fact, we can represent the FAME in-scan and cross-scan observables as Δu and Δv values, where Δu is the O–C value in the in-scan direction and Δv is the O–C value in the cross-scan direction.

For the computation of the astrometric partials, we establish a local coordinate system on the sky, [$\mathbf{r}$,$\mathbf{d}$], at the position $\mathbf{p}$ of the star being observed. The basis vectors $\mathbf{r}$ and $\mathbf{d}$ are in the plane of the sky and point in the directions of increasing right ascension and increasing declination, respectively. The [$\mathbf{r}$,$\mathbf{d}$] system can be established in a manner completely analogous to that used for the [$\mathbf{u}$,$\mathbf{v}$] system:

$$\begin{aligned} \mathbf{r} &= \frac{\mathbf{n}\times\mathbf{p}}{|\mathbf{n}\times\mathbf{p}|} \\ \mathbf{d} &= -\mathbf{r}\times\mathbf{p} \end{aligned} \tag{2}$$

Here, $\mathbf{n}$ is the unit vector toward the ICRS north celestial pole. This equation is the analog to eqn. (11) in [1]. These vectors are trivial to compute:

$$\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{p} = \begin{pmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{pmatrix}$$

$$\mathbf{n}\times\mathbf{p} = \begin{pmatrix} -\cos\delta\sin\alpha \\ \cos\delta\cos\alpha \\ 0 \end{pmatrix} \qquad |\mathbf{n}\times\mathbf{p}| = \cos\delta \qquad \mathbf{d} = \begin{pmatrix} -\sin\delta\cos\alpha \\ -\sin\delta\sin\alpha \\ \cos\delta \end{pmatrix} \tag{3}$$

where α and δ are the RA and Dec of the star. Whether we should use the apparent position of the star (denoted $\mathbf{p}'$ in [1]) or the catalog position of the star ($\mathbf{p}_0$) is discussed below. There are, of course, as many [$\mathbf{r}$,$\mathbf{d}$] systems as there are stars in the field; properly, we should write $[\mathbf{r},\mathbf{d}]_{ij}$ to denote the coordinate system for star ij (following the notation of [1]), but for the sake of simplicity of notation we will omit the subscript.

For each star ij we now have two 2D coordinate systems, [$\mathbf{u}$,$\mathbf{v}$] and [$\mathbf{r}$,$\mathbf{d}$], for the local patch of sky. The transformation between these two systems is just

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mathbf{u}\cdot\mathbf{r} & \mathbf{u}\cdot\mathbf{d} \\ \mathbf{v}\cdot\mathbf{r} & \mathbf{v}\cdot\mathbf{d} \end{pmatrix} \begin{pmatrix} r \\ d \end{pmatrix} + \begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} \tag{4}$$

where (u_{ij}, v_{ij}) are the coordinates of star ij in the [$\mathbf{u}$,$\mathbf{v}$] system. The difference in the two systems' origins is unimportant for present purposes, as the partial derivatives relating variations in one set of coordinates with those in the other set must be simply:

$$\frac{\partial u}{\partial r} = \mathbf{u}\cdot\mathbf{r} \qquad \frac{\partial u}{\partial d} = \mathbf{u}\cdot\mathbf{d} \qquad \frac{\partial v}{\partial r} = \mathbf{v}\cdot\mathbf{r} \qquad \frac{\partial v}{\partial d} = \mathbf{v}\cdot\mathbf{d} \tag{5}$$

Hence the partial derivatives of the O–C's in u and v with respect to a position offset of the star in RA and Dec are

$$\frac{\partial u}{\partial(\Delta\alpha\cos\delta)} = \mathbf{u}\cdot\mathbf{r} \qquad \frac{\partial u}{\partial(\Delta\delta)} = \mathbf{u}\cdot\mathbf{d} \qquad \frac{\partial v}{\partial(\Delta\alpha\cos\delta)} = \mathbf{v}\cdot\mathbf{r} \qquad \frac{\partial v}{\partial(\Delta\delta)} = \mathbf{v}\cdot\mathbf{d} \tag{6}$$

The two cordinate systems are each orthonormal, but they are of opposite handedness if we consider their two z axes to point in the same direction. We therefore have $\mathbf{u}\cdot\mathbf{r} = -\mathbf{v}\cdot\mathbf{d}$ and $\mathbf{u}\cdot\mathbf{d} = \mathbf{v}\cdot\mathbf{r}$. Also $(\mathbf{u}\cdot\mathbf{r})^2 + (\mathbf{v}\cdot\mathbf{r})^2 = (\mathbf{u}\cdot\mathbf{d})^2 + (\mathbf{v}\cdot\mathbf{d})^2 = 1$.

To form the partials for proper motion, the above expressions are multiplied by $t - t_0$, the difference between the time of observation and the selected reference epoch. Partials for higher-order terms in the star's motion are formed by multiplying the above expressions by $(t - t_0)^n$.

If we use the apparent position of the star, $\mathbf{p}'$, to form $\mathbf{r}$ and $\mathbf{d}$ in eqn. (2), the [$\mathbf{r}$,$\mathbf{d}$] system will change orientation with respect to the ICRS as the star moves due to aberration and proper motion. For the vast majority of stars this effect is quite small, but near the celestial pole it blows up. In the extreme case, for a star close enough to the pole that the pole is *within* its aberrational ellipse, the [$\mathbf{r}$,$\mathbf{d}$] system would rotate through 360° over the course of a year! Somewhat farther from the pole there will still be troublesome annual oscillations of the [$\mathbf{r}$,$\mathbf{d}$] orientation. Needless to say, when combining observations from different epochs this situation is unacceptable. Therefore it seems that we should use the catalog position of the star, $\mathbf{p}_0$, in forming the [$\mathbf{r}$,$\mathbf{d}$] system. (Again, the point of origin of the system is irrelevant; only the orientation matters.) As the data analysis progresses and the catalog is improved, this system will evolve very slightly for each star. However, the [$\mathbf{r}$,$\mathbf{d}$] system would remain fixed for each star until a catalog update, and that is, I believe, what we want for the data analysis.

There is a slight swindle in the above: the [$\mathbf{u}$,$\mathbf{v}$] and [$\mathbf{r}$,$\mathbf{d}$] systems are not exactly coplanar, since the direction of the aperture boresight and direction of the star are not parallel. For stars near the edge of a degree-wide field this may affect the partials at the 4×10^{-5} level, acceptable for O–C's of ~0.1 arcsec. The O–C's will get smaller as the catalog improves.

2.3 Partials for Parallax

The development of the parallax partial is similar to that for position and proper motion. We take FAME's position vector with respect to the solar system barycenter, $\mathbf{F}$, and project it onto the sky at the star's position $\mathbf{p}'$.[1] Both $\mathbf{p}'$ and $\mathbf{F}$ are in the ICRS; $\mathbf{p}'$ is a unit vector and $\mathbf{F}$ is expressed in AU. The projection of $\mathbf{F}$ onto the plane of the sky is

$$\mathbf{f} = \mathbf{F} - (\mathbf{F} \cdot \mathbf{p}')\mathbf{p}' \tag{7}$$

The position of FAME with respect to the solar system barycenter shifts the apparent position of the star in the direction $-\mathbf{f}$. The star's parallax in arcsec, x, is just the scaling factor for $-\mathbf{f}$. The partial derivatives of the O–C's in u and v with respect to x must therefore simply be the direction cosines of $-\mathbf{f}$ with respect to the $\mathbf{u}$ and $\mathbf{v}$ axes:

$$\frac{\partial u}{\partial x} = -\mathbf{f} \cdot \mathbf{u} \qquad\qquad \frac{\partial v}{\partial x} = -\mathbf{f} \cdot \mathbf{v} \tag{8}$$

Here again, there is a small error due to the fact that $\mathbf{f}$ is not exactly in the [$\mathbf{u}$,$\mathbf{v}$] plane.

[1] I believe that this version of $\mathbf{p}'$ should include proper motion and parallax to the extent known, but *not* aberration or light bending. The difference in the final partial can amount to $\sim 10^{-4}$ and may not be critical, but the point needs further thought.

3 Ecliptic Coordinates

Since FAME's scanning strategy involves a spin axis that precesses along a circle centered on the Sun, the ecliptic coordinate system could be considered the "natural" system for FAME data analysis. We expect that the errors and covariances for the astrometric parameters that will come out of the solution will make the most sense, in the aggregate, when expressed in the ecliptic system. Because FAME's spin axis never comes close to the ecliptic pole, some degeneracies in rotational transformations are also avoided when using ecliptic coordinates.

The true ecliptic, defined by the plane of the instantaneous position and velocity vectors of the Earth, undergoes small-scale oscillations due to lunar and planetary perturbations. For some purposes it is convenient to define a "mean ecliptic" with the short-term periodic motions filtered out; it is basically a long-term average of the true ecliptic's motion. In these days of numerically integrated planetary motions, the distinction between "short term" and "long term" motions can be problematic. In any event, the orientation of the ecliptic is currently changing at a secular rate of about 0.5 arcsec/year with respect to an inertial system.

Given that very slow rate of change, it would be inadvisable to attempt to use any kind of moving ecliptic for FAME data analysis. We do not need the complications of dealing with a coordinate system that is itself rotating. What we can do is use the fixed ecliptic of J2000.0 (or the fixed ecliptic of an epoch in the middle of the observations) as an adequate approximation to the moving ecliptic.

For an arbitrary vector $\mathbf{V}$ expressed in the ICRS, its ecliptic equivalent is given by a rotation about the x-axis:

$$\mathbf{V}_{\text{ecliptic}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & \sin\epsilon \\ 0 & -\sin\epsilon & \cos\epsilon \end{pmatrix} \mathbf{V}_{\text{ICRS}} \tag{9}$$

where ϵ is the obliquity of the ecliptic in the ICRS. The current best estimate for the value of the mean obliquity at J2000.0 is $\epsilon = 23^\circ\, 26'\, 21.4059''$ as given in [4]. The location of the north ecliptic pole in the ICRS is then at RA $= 18^{\mathrm{h}}$, Dec $= 90^\circ{-}\epsilon$, with unit vector

$$\mathbf{n}_e = \begin{pmatrix} 0 \\ -\sin\epsilon \\ \cos\epsilon \end{pmatrix}$$

The angular coordinates in the ecliptic system are designated ecliptic longitude, λ, and ecliptic latitude, β, and are measured in the same sense as RA and Dec. As noted above, the two systems share the x-axis so the zero point of the azimuthal coordinate is the equinox for both. The location of the north celestial pole in the ecliptic system is at $\lambda = 90^\circ$, $\beta = 90^\circ{-}\epsilon$.

Changing to an ecliptic system for the pipeline calculations is quite simple. There are two options.

Option A If we want to put only the astrometric parameters (and their errors and covariances) in the ecliptic system, then all we have to do is substitute $\mathbf{n}_e$ for $\mathbf{n}$ in eqn. (2). (The explicit representation of the vectors in eqn. (3) cannot be used for this case, however.) We will call the resulting system $[\mathbf{r}_e, \mathbf{d}_e]$, where $\mathbf{r}_e$ points in the direction of increasing ecliptic longitude and $\mathbf{d}_e$ points in the direction of increasing ecliptic latitude. It should be noted that for this case the subscript-e vectors relate to the ecliptic system but are themselves expressed in ICRS coordinates. In the ecliptic system, the star position offsets (and proper motion) are then expressed in terms of $\Delta\lambda \cos\beta$ and $\Delta\beta$, so eqn. (6) becomes

$$\frac{\partial u}{\partial(\Delta\lambda\cos\beta)} = \mathbf{u}\cdot\mathbf{r}_e \qquad \frac{\partial u}{\partial(\Delta\beta)} = \mathbf{u}\cdot\mathbf{d}_e \qquad \frac{\partial v}{\partial(\Delta\lambda\cos\beta)} = \mathbf{v}\cdot\mathbf{r}_e \qquad \frac{\partial v}{\partial(\Delta\beta)} = \mathbf{v}\cdot\mathbf{d}_e \tag{10}$$

Equations (7) and (8) for parallax are untouched.

Option B If we want to shift the entire set of pipeline calculations, including those for spacecraft attitude, to the ecliptic system, a little more work — but not much more — is required. The input star catalog and the apparent place calculations can remain in the ICRS, as can the input file of attitude quaternions. We simply have to put the following vectors through the rotation given in eqn. (9):

$\mathbf{p}_0$	unit vector toward catalog position of star (transform *after* apparent place is computed)
$\mathbf{p}'$	unit vector toward apparent place of star
$\mathbf{F}$	position vector of FAME wrt solar system barycenter, returned by apparent place calculation
$\mathbf{x}$,$\mathbf{y}$,$\mathbf{z}$	instantaneous directions of FAME's spacecraft-fixed axes
$\boldsymbol{\omega}$	instantaneous FAME rotation axis

All other vectors and scalars are derived from these. We can use eqns. (2) and (3) as is, with the understanding that λ and β have effectively replaced α and δ. We do not use $\mathbf{n}_e$, because $\mathbf{n} = (0, 0, 1)$ will point to the ecliptic pole. If we call the resulting system $[\mathbf{r}_e, \mathbf{d}_e]$ (for consistency with Option A), then the partials of the astrometric parameters are given by eqn. (10). Equations (7) and (8) for parallax remain valid.

Note that if the ecliptic transformations are included in the pipeline (either option), we can easily revert to the ICRS simply by setting the obliquity ϵ to zero. That is, the obliquity value can act as a "switch" to select either ICRS or ecliptic calculations.

One other piece of unfinished business remains. Since the FAME star catalog will remain stored in ICRS coordinates, we need to be able to transform corrections to the astrometric parameters, expressed in the ecliptic system, back to the ICRS. This transformation would need to be done either to update the catalog or, during debugging, to check whether the pipeline had recovered deliberately introduced catalog errors. This can be treated as a transformation of differential quantities, local to the small area of sky around each star, from the ecliptic-based [$\mathbf{r}_e$,$\mathbf{d}_e$] system to the ICRS-based [$\mathbf{r}$,$\mathbf{d}$] system:

$$\begin{pmatrix} \Delta\alpha\cos\delta \\ \Delta\delta \end{pmatrix} = \begin{pmatrix} \mathbf{r}\cdot\mathbf{r}_e & \mathbf{r}\cdot\mathbf{d}_e \\ \mathbf{d}\cdot\mathbf{r}_e & \mathbf{d}\cdot\mathbf{d}_e \end{pmatrix} \begin{pmatrix} \Delta\lambda\cos\beta \\ \Delta\beta \end{pmatrix} \tag{11}$$

which obviously requires that all four vectors, $\mathbf{r}$, $\mathbf{d}$, $\mathbf{r}_e$, and $\mathbf{d}_e$, be computed, even though only one pair is used for the main part of the calculations. For Option B we can obtain $\mathbf{r}$ and $\mathbf{d}$ if we set

$$\mathbf{n} = \begin{pmatrix} 0 \\ \sin\epsilon \\ \cos\epsilon \end{pmatrix}$$

then use eqn. (2). Here, $\mathbf{n}$ is the direction of the north celestial pole in the ecliptic system.

References

[1] Kaplan, G. H., 2001, "Algorithms for Rapidly Computing the FAME Observing Sequence", available at http://fame.usno.navy.mil/DocuShare/dscgi/ds.py/Get/File-539/OBS_SEQ.PDF

[2] Kaplan, G. H., 2001, "Data Analysis Plan for the Full-sky Astrometric Mapping Explorer (FAME)", USNO-FM006, available at http://fame.usno.navy.mil/DocuShare/dscgi/ds.py/Get/File-852/USNO-FM006.doc

[3] Makarov, V. V., Untitled handwritten notes of 6/8/01

[4] Fukushima, T. 2002, Highlights of Astronomy, 12A (in press).